

# POSITIVE DEFINITE \*-SPHERICAL FUNCTIONS, PROPERTY (T), AND $C^*$ -COMPLETIONS OF GELFAND PAIRS

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**ABSTRACT.** The study of existence of a universal  $C^*$ -completion of the  $*$ -algebra canonically associated to a Hecke pair was initiated by Hall, who proved that the Hecke algebra associated to  $(\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{Z}_p))$  does not admit a universal  $C^*$ -completion. Kaliszewski, Landstad and Quigg studied the problem by placing it in the framework of Fell-Rieffel equivalence, and along the way highlighted the role of other  $C^*$ -completions. In the case of the pair  $(\mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Z}_p))$  for  $n \geq 3$  we show, invoking property (T) of  $\mathrm{SL}_n(\mathbb{Q}_p)$ , that the  $C^*$ -completion of the  $L^1$ -Banach algebra and the corner of  $C^*(\mathrm{SL}_n(\mathbb{Q}_p))$  determined by the subgroup are distinct. This complements the similar result for  $n = 2$  due to the second named author, and provides a proof for a question raised by Kaliszewski, Landstad and Quigg.

## 1. INTRODUCTION

The work of Bost and Connes on a  $C^*$ -dynamical system with deep connections to class field theory brought to attention  $C^*$ -algebras associated to Hecke pairs, [2]. A (discrete) Hecke pair  $(G, \Gamma)$  consists of a group  $G$  with a subgroup  $\Gamma$  such that every double coset  $\Gamma g \Gamma$  contains finitely many left cosets, for every  $g$  in  $G$ . The associated Hecke algebra  $\mathcal{H}(G, \Gamma)$  is a convolution  $*$ -algebra of complex-valued functions on the space of double cosets, see for example [8]. Hall initiated the study of  $C^*$ -completions of  $\mathcal{H}(G, \Gamma)$  in connection to asking whether there is a category equivalence between the nondegenerate  $*$ -representations of  $\mathcal{H}(G, \Gamma)$  and the unitary representations of  $G$  generated by their  $\Gamma$ -fixed vectors, [6]. The question was motivated by the well-known fact that unitary representations of a group are in bijective correspondence with the nondegenerate  $*$ -representations of the group algebra.

While proposing a condition that would yield an affirmative answer to the question, Hall showed at the same time that the equivalence could not hold in all generality. Indeed, she showed that  $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{Z}_p))$  does not admit an enveloping  $C^*$ -algebra, [6]. The proof was based on a careful description, via the Satake isomorphism, of this algebra as a polynomial algebra in one variable. This showed that there are elements of  $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{Z}_p))$  which have arbitrarily large norms with respect to  $*$ -representations.

Shortly afterwards there were two related developments. In one of them Tzanev, motivated by amenability in connection with Hecke pairs, and drawing on work of Schlichting [14], noticed that to every discrete Hecke pair  $(G, \Gamma)$  one can associate in a canonical way a topological pair  $(\overline{G}, \overline{\Gamma})$  having isomorphic Hecke algebra, [15]. In another development, Kaliszewski, Landstad and Quigg placed the study of Hall's correspondence in the framework of Fell-Rieffel equivalence, [7]. Both approaches investigated  $C^*$ -completions of  $\mathcal{H}(G, \Gamma)$  by looking at the corner  $p_0 C_c(\overline{G}) p_0$  arising from the self-adjoint projection  $p_0$  given by the characteristic function of the compact open subgroup  $\overline{\Gamma}$  of  $\overline{G}$ .

One question raised in [7] was when a certain canonical surjection from  $C^*(p_0 L^1(\overline{G}) p_0)$  onto  $p_0 C^*(\overline{G}) p_0$  could fail to be injective. It is stated in [7, page 677] that the injectivity fails for the

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pair  $(\mathrm{PSL}_3(\mathbb{Q}_p), \mathrm{PSL}_3(\mathbb{Z}_p))$ , according to a personal communication by Tzanev. However, to our knowledge, no proof of this claim has been published. The result showing that injectivity of the canonical map above fails for the pair  $(\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{Z}_p))$  is due to the second named author, [9].

Our main contribution here is to prove that the canonical surjection from  $C^*(p_0 L^1(\mathrm{SL}_n(\mathbb{Q}_p)) p_0)$  onto  $p_0 C^*(\mathrm{SL}_n(\mathbb{Q}_p)) p_0$  is not injective for all  $n \geq 3$ . The proof differs from the case of  $n = 2$  in [9] in that it appeals to property (T). After the second named author's talk on the case  $n = 2$  at a NordForsk conference on the Farøe Islands in 2012, a discussion with M. Landstad revealed that the obstruction to injectivity in the case  $n = 3$  would be property (T).

Thus our initial investigations concentrated on property (T) for a Hecke pair  $(G, \Gamma)$ . This notion was already introduced in [16, page 24], where it was claimed that one could easily show that property (T) of a Hecke pair  $(G, \Gamma)$  was equivalent to property (T) of the locally compact group  $\overline{G}$ , and that all other characterisations of property (T) for groups could be translated to Hecke pairs. However, no details were given for any of these two claims. Here we show that the first claim is valid, though the arguments do require a careful analysis; this material forms the content of appendix A. Regarding the second claim, we found that one characterisation does not necessarily translate to Hecke pairs. Namely, it is known that for a locally compact group  $G$  with property (T), the trivial representation is isolated in its natural Fell topology. However, in taking a compact open subgroup  $U$  and forming the Hecke pair  $(G, U)$ , the trivial representation of  $C^*(L^1(G, U))$  need not be isolated in the natural hull-kernel topology on  $(C^*(L^1(G, U)))^\wedge$ . In fact, this is precisely what happens when  $(G, U)$  is  $(\mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Z}_p))$  for  $n \geq 3$ .

As a consequence of this distinction, the trivial representation is isolated in the corner  $p_0 C^*(\mathrm{SL}_n(\mathbb{Q}_p)) p_0$  with its hull-kernel topology, but not in  $(C^*(L^1(G, U)))^\wedge$  with its hull-kernel topology, see Proposition 5.1 and Theorem 5.2. The crucial technical tool we use to reach this conclusion is a description of all spherical functions for the pair  $(\mathrm{SL}_n(\mathbb{Q}_p), \mathrm{SL}_n(\mathbb{Z}_p))$  for  $n \geq 3$  due to Satake, [13]. The spherical functions are known to correspond to characters on the abelian Banach algebra  $L^1(G, U)$  associated to the Gelfand pair  $(G, U)$ . Since our interest is in  $*$ -representations of  $\mathcal{H}(G, U)$ , we need to consider a particular class of spherical functions, which we call  $*$ -spherical functions. We show that the distinction in isolation of the trivial representation in the natural topologies on  $C^*(L^1(G, U)) = C^*(p_0 L^1(G) p_0)$  and  $p_0 C^*(G) p_0$ , respectively, is due to the fact that not all bounded  $*$ -spherical functions on  $\mathcal{H}(G, U)$  are positive definite. Indeed, we show that for an arbitrary Gelfand pair  $(G, U)$ , the canonical surjection from  $C^*(L^1(G, U))$  onto  $p_0 C^*(G) p_0$  is injective precisely when all bounded  $*$ -spherical functions on  $\mathcal{H}(G, U)$  are positive definite, cf. Theorem 3.1.

Another consequence of our investigations is that we can show that for a Gelfand pair  $(G, U)$ , a necessary and sufficient condition for the existence of the universal  $C^*$ -completion of  $\mathcal{H}(G, U)$  is that all  $*$ -spherical functions are bounded, see Theorem 3.2. This behaviour can perhaps be interpreted as a feature common to the group algebra in the case of abelian  $G$ , since by taking  $U$  to be the trivial subgroup, the spherical functions are characters on  $G$ , and hence automatically positive definite and bounded.

We thank Magnus Landstad for indicating that property (T) would be the right notion to consider in order to investigate injectivity of the canonical map when  $n \geq 3$ .

## 2. GENERALITIES ABOUT SPHERICAL FUNCTIONS

We begin by introducing some notation and recalling terminology. Suppose that  $(G, \Gamma)$  is a Hecke pair, by which we mean either a discrete pair or a pair formed of a topological group with an open subgroup such that every double coset contains finitely many left cosets. With  $L(g)$  denoting the number of left cosets in  $\Gamma g \Gamma$  for  $g \in G$ , the map  $\Delta(g) = L(g)/L(g^{-1})$  is a group

homomorphism of  $G$  into  $\mathbb{Q}^+$ . A function  $f : G \rightarrow \mathbb{C}$  is  $\Gamma$ -biinvariant if  $f(xg) = f(gx) = f(g)$  for all  $g \in G$  and  $x \in \Gamma$ . The *Hecke algebra*  $\mathcal{H}(G, \Gamma)$  associated to  $(G, \Gamma)$  is the  $*$ -algebra of  $\Gamma$ -biinvariant complex valued functions on  $G$  which are finitely supported when viewed on  $\Gamma \backslash G / \Gamma$ ; the convolution and involution in  $\mathcal{H}(G, \Gamma)$  are defined by

$$(2.1) \quad (f_1 * f_2)(g) = \sum_{h\Gamma \in G/\Gamma} f_1(h) f_2(h^{-1}g), \text{ and}$$

$$(2.2) \quad f(g)^* = \Delta(g^{-1}) \overline{f(g^{-1})},$$

for  $f_1, f_2, f \in \mathcal{H}(G, \Gamma)$ , where the sum in the convolution formula is taken over representatives for the left coset space.

One can define an  $L^1$ -norm on  $\mathcal{H}(G, \Gamma)$  for any Hecke pair  $(G, \Gamma)$  by

$$(2.3) \quad \|f\|_1 = \sum_{\Gamma g \Gamma \in \Gamma \backslash G / \Gamma} |f(\Gamma g \Gamma)| L(g),$$

and take the corresponding completion  $L^1(G, \Gamma)$ , which is a Banach  $*$ -algebra.

An example of a Hecke pair arises when  $G$  is a locally compact topological group and  $U$  is a compact open subgroup. This type of example is generic, cf. [15], see Section 3 for more details. Given such  $(G, U)$ , the Hecke algebra can also be defined as the space of complex valued, compactly supported functions on  $U \backslash G / U$  which are  $U$ -biinvariant; the convolution and the involution are defined as in (2.1) and (2.2), where  $\Delta$  is the modular function of  $G$ . Let  $\mu$  be a left Haar measure on  $G$  normalized by  $\mu(U) = 1$ , and consider the Banach  $*$ -algebra  $L^1(G)$  endowed with its convolution

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) d\mu(h)$$

for  $f_1, f_2 \in L^1(G)$  and involution defined as above. The subspace  $L^1(U \backslash G / U)$  of  $U$ -biinvariant functions in  $L^1(G)$  becomes a closed subalgebra of  $L^1(G)$ . Let  $p_0$  be the self-adjoint projection in  $L^1(G)$  (and in  $C_c(G)$ ) determined by  $\chi_U$ . Then  $\mathcal{H}(G, U) = p_0 C_c(G) p_0$  and there are canonical isomorphisms  $L^1(U \backslash G / U) \cong L^1(G, U) \cong p_0 L^1(G) p_0$ .

Recall that when  $G$  is a *unimodular* locally compact group and  $U$  a compact open subgroup, then  $(G, U)$  is a *Gelfand pair* if  $L^1(U \backslash G / U)$  is commutative. The notion of a spherical function for a Gelfand pair is well-known, cf. for example [5] or the expository article [3]. Here we follow [13, §5.2].

**Definition 2.1.** Let  $G$  be a unimodular locally compact group and  $U$  a compact subgroup. A continuous function  $\omega : G \rightarrow \mathbb{C}$  is a spherical function of  $G$  relative to  $U$  if the following conditions are satisfied:

- i)  $\omega(e) = 1$ .
- ii)  $\omega(u_1 g u_2) = \omega(g)$  for all  $g \in G$  and  $u_1, u_2 \in U$ .
- iii)  $\int_U \omega(g_1 u g_2) d\mu(u) = \omega(g_1) \omega(g_2)$ , for all  $g_1, g_2 \in G$ .

If  $U$  is a compact open subgroup of  $G$ , then we call a function  $\omega$  as in Definition 2.1 a *spherical function* for  $\mathcal{H}(G, U)$ . Assuming i) and ii), condition iii) is equivalent to the following:

- iv) For each  $f \in \mathcal{H}(G, U)$  there exists  $\lambda_f \in \mathbb{C}$  such that  $f * \omega = \lambda_f \omega$ .

Given a spherical function  $\omega$  for  $\mathcal{H}(G, U)$ , one associates a homomorphism  $\tau_\omega : \mathcal{H}(G, U) \rightarrow \mathbb{C}$  defined by

$$(2.4) \quad \tau_\omega(f) := \int_G f(g) \omega(g^{-1}) d\mu(g).$$

If  $U$  is moreover open, then every non-trivial homomorphism of  $\mathcal{H}(G, U)$  into  $\mathbb{C}$  arises in this way (a result attributed in [13] to T. Tamagawa). Thus  $\omega \longleftrightarrow \tau_\omega$  establishes a bijective correspondence between spherical functions and homomorphisms of  $\mathcal{H}(G, U)$  onto  $\mathbb{C}$ . This bijective correspondence holds for non-trivial homomorphisms of  $L^1(G, U)$  into  $\mathbb{C}$  when one considers bounded spherical functions.

**Proposition 2.2.** *Let  $(G, U)$  be a Gelfand pair and  $\omega$  a spherical function. The homomorphism  $\tau_\omega$  extends to a homomorphism  $L^1(G, U) \rightarrow \mathbb{C}$  if and only if  $\omega$  is bounded with  $|\omega(\cdot)| \leq 1$ .*

**Proof:** For the forward direction, assume that  $\tau_\omega$  extends to  $L^1(G, U)$ . Thus  $|\tau_\omega(f)| \leq \|f\|_1$  for all  $f \in L^1(G, U)$ . For  $f = \chi_{UgU}$  we have

$$\begin{aligned} |\tau_\omega(\chi_{UgU})| &= \left| \int_G \chi_{UgU}(h) \omega(h^{-1}) d\mu \right| \\ &= \left| \int_{UgU} \chi_{UgU}(h) \omega(h^{-1}) d\mu \right| \\ &= |\omega(g^{-1})| \cdot \left| \int_{UgU} \chi_{UgU}(h) d\mu \right| \\ &= |\omega(g^{-1})| \cdot \|\chi_{UgU}\|_1. \end{aligned}$$

Hence  $|\omega(g^{-1})| \leq 1$ , and since  $g$  was arbitrary, it follows that  $|\omega(\cdot)| \leq 1$ .

The converse direction follows because for each  $f \in \mathcal{H}(G, U)$  we have

$$\begin{aligned} |\tau_\omega(f)| &= \left| \int_G f(g) \omega(g^{-1}) d\mu \right| \leq \int_G |f(g)| |\omega(g^{-1})| d\mu \\ &\leq \int_G |f| d\mu = \|f\|_1. \end{aligned}$$

□

In the next result we identify which spherical functions give rise to  $*$ -homomorphisms of  $\mathcal{H}(G, U)$ .

**Proposition 2.3.** *Let  $(G, U)$  be a Gelfand pair and  $\omega$  a spherical function. The homomorphism  $\tau_\omega$  is a  $*$ -homomorphism if and only if  $\omega(g^{-1}) = \overline{\omega(g)}$  for all  $g \in G$ .*

**Proof:** Let  $\tilde{\omega}$  be the function on  $G$  defined by  $\tilde{\omega}(g) = \overline{\omega(g^{-1})}$ . It is not difficult to see that  $\tilde{\omega}$  is also a spherical function. For every  $f \in \mathcal{H}(G, U)$  we have

$$\begin{aligned} \overline{\tau_\omega(f^*)} &= \overline{\int_G f^*(g) \omega(g^{-1}) d\mu} = \overline{\int_G \overline{f(g^{-1})} \omega(g^{-1}) d\mu} \\ &= \int_G f(g^{-1}) \overline{\omega(g^{-1})} d\mu = \int_G f(g) \overline{\omega(g)} d\mu \\ &= \int_G f(g) \tilde{\omega}(g^{-1}) d\mu = \tau_{\tilde{\omega}}(f). \end{aligned}$$

Hence, if  $\tau_\omega$  is a  $*$ -homomorphism, then  $\tau_\omega(f) = \overline{\tau_\omega(f^*)} = \tau_{\tilde{\omega}}(f)$ , so  $\tau_\omega = \tau_{\tilde{\omega}}$ . Since  $\omega \longleftrightarrow \tau_\omega$  is a bijective correspondence,  $\omega = \tilde{\omega}$ .

Conversely, if  $\omega = \tilde{\omega}$ , then  $\tau_\omega(f) = \tau_{\tilde{\omega}}(f) = \overline{\tau_\omega(f^*)}$  and therefore  $\tau_\omega$  is a  $*$ -homomorphism.

□

**Definition 2.4.** Let  $G$  be a unimodular locally compact group and  $U$  a compact open subgroup. A spherical function  $\omega$  for  $\mathcal{H}(G, U)$  that satisfies  $\omega(g^{-1}) = \overline{\omega(g)}$  for all  $g \in G$  will be called a *\*-spherical function*.

According to Proposition 2.3, the \*-spherical functions for a Gelfand pair are precisely those for which the associated map  $\tau_\omega$  is a \*-homomorphism.

**Proposition 2.5.** Suppose that  $\{\omega_n\}_{n \in \mathbb{N}}$  is a sequence of spherical functions and  $\omega$  a spherical function for  $\mathcal{H}(G, U)$  such that  $|\omega_n(\cdot)| \leq 1$  and  $|\omega(\cdot)| \leq 1$ . If  $\omega_n \rightarrow \omega$  uniformly on compact subsets of  $G$ , then  $\tau_{\omega_n} \rightarrow \tau_\omega$  in the weak\*-topology, as linear functionals of  $L^1(G, U)$ .

**Proof:** Note that Proposition 2.2 guarantees that  $\tau_\omega$  and  $\tau_{\omega_n}$ , for each  $n \geq 1$ , extend to homomorphisms of  $L^1(G, U)$  to  $\mathbb{C}$ . Let  $f \in L^1(G, U)$  and  $\epsilon > 0$ . Since  $f$  can be viewed as a function in  $L^1(G)$ , we can choose a sufficiently large compact set  $K \subseteq G$  such that  $\int_{G \setminus K} f \, d\mu < \epsilon/4$ . Choose  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$  we have  $\|(\omega_n - \omega)|_{K^{-1}}\|_\infty < \epsilon/2\|f\|_1$ . Then for  $n \geq n_0$  we have

$$\begin{aligned} |\tau_{\omega_n}(f) - \tau_\omega(f)| &\leq \int_{G \setminus K} |f(g)| |\omega_n(g^{-1}) - \omega(g^{-1})| \, d\mu(g) \\ &\quad + \int_K |f(g)| \, d\mu(g) \cdot \|(\omega_n - \omega)|_{K^{-1}}\|_\infty \\ &\leq \frac{2\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Given a Gelfand pair  $(G, U)$ , a spherical function  $\omega : G \rightarrow \mathbb{C}$  is *positive definite* if for any  $n \geq 1$ , any finite subset  $\{s_1, \dots, s_m\}$  of  $G$  and any collection  $\{z_1, \dots, z_m\}$  of complex numbers, one has

$$\sum_{j,k=1}^m \omega(s_j^{-1} s_k) \overline{z_j} z_k \geq 0,$$

see for example [3]. We mention that a different formulation of this notion appears in [13, Appendix II]. Positive definite spherical functions for a Gelfand pair  $(G, U)$  appear naturally within the context of unitary representations. In the general theory of Hecke algebras a unitary representation of  $G$  on a Hilbert space which is generated by its  $U$ -fixed vectors always gives rise to a nondegenerate \*-representation of the Hecke algebra  $\mathcal{H}(G, U)$ . The converse, whether every nondegenerate \*-representation of  $\mathcal{H}(G, U)$  arises in this way from a unitary representation, is not always true. Hall proposed a positivity condition for a certain inner product as a sufficient condition for a category equivalence of the two classes of representations, [6, Theorem 3.25]. The category equivalence holds true when certain  $C^*$ -completions of the Hecke algebra are the same, see [7, Corollaries 6.11(ii) and 6.19]. In the case of Gelfand pairs, bounded positive definite spherical functions  $\omega$  are precisely those for which the associated \*-representation  $\tau_\omega$  arises from a unitary representation of  $G$ . Before we indicate the relation between positive definite spherical functions and  $C^*$ -completions of Hecke algebras, we will first show that positive definite spherical functions are always bounded and \*-spherical. The result is well-known, cf. for example [13, Appendix II], but we include a proof for convenience.

**Proposition 2.6.** Let  $(G, U)$  be a Gelfand pair. Every positive definite spherical function is necessarily bounded and \*-spherical.

**Proof:** Let  $\omega$  be a positive definite spherical function. Consider the subsets  $\{g, e\}$  of  $G$  and  $\{z_1, z_2\}$  of  $\mathbb{C}$ . By definition of positive definiteness, and using the fact that  $\omega(e) = 1$ , we have

$$|z_1|^2 + \omega(g^{-1})\overline{z_1}z_2 + \omega(g)z_1\overline{z_2} + |z_2|^2 \geq 0.$$

Taking  $z_1 = z_2 = 1$  it follows that  $\omega(g^{-1}) + \omega(g) \geq 0$ , from which we conclude that the imaginary parts of these elements satisfy  $\text{Im}(\omega(g^{-1})) = -\text{Im}(\omega(g))$ . Taking  $z_1 = i$  and  $z_2 = 1$  it follows that  $2 - i\omega(g^{-1}) + i\omega(g) \geq 0$ , which means that  $-\omega(g^{-1}) + \omega(g)$  is purely imaginary. Hence, the real parts of the elements  $\omega(g^{-1})$  and  $\omega(g)$  must be the same. Thus we have shown that  $\omega(g^{-1}) = \overline{\omega(g)}$ . Since this is true for any  $g \in G$ ,  $\omega$  is  $*$ -spherical.

Taking  $z_1 = \omega(g^{-1})$  and  $z_2 = -1$  we obtain  $|\omega(g^{-1})|^2 - |\omega(g^{-1})|^2 - \omega(g)\omega(g^{-1}) + 1 \geq 0$ . Since  $\omega$  is  $*$ -spherical it follows that  $1 \geq |\omega(g)|^2$ . As this holds for all  $g \in G$ ,  $\omega$  is bounded.  $\square$

### 3. THE CANONICAL SURJECTION FOR A HECKE PAIR AND $*$ -SPHERICAL FUNCTIONS

To motivate our first result, we recall that for a (discrete) Hecke pair  $(G, \Gamma)$ , the convolution and involution on the associated Hecke algebra  $\mathcal{H}(G, \Gamma)$  are as given in equations (2.1) and (2.2). It was shown in [15], see also [7], that there is a unique pair  $(\overline{G}, \overline{\Gamma})$  where  $\overline{G}$  is a totally disconnected locally compact group,  $\overline{\Gamma}$  is a compact open subgroup, there is a canonical embedding  $\iota : G \rightarrow \overline{G}$  such that  $\iota(G)$  is dense in  $\overline{G}$ ,  $\iota(\Gamma)$  is dense in  $\overline{\Gamma}$ , and  $\iota^{-1}(\overline{\Gamma}) = \Gamma$  (more precisely, the uniqueness is achieved by passing to the reduction  $(G_r, \Gamma_r)$  of  $(G, \Gamma)$ ). We refer to  $(\overline{G}, \overline{\Gamma})$  as the Schlichting completion of  $(G, \Gamma)$ .

Let  $p_0$  denote the self-adjoint projection  $\chi_{\overline{\Gamma}}$  in  $C_c(\overline{G})$ . There are canonical isomorphisms  $\mathcal{H}(G, \Gamma) \cong \mathcal{H}(\overline{G}, \overline{\Gamma}) \cong p_0 C_c(\overline{G}) p_0$ . Upon completion with respect to the norm from (2.3), one obtains an isomorphism  $L^1(G, \Gamma) \cong p_0 L^1(\overline{G}) p_0$ .

The canonical surjection for the Hecke pair  $(G, \Gamma)$  alluded to in the title of the section is the  $*$ -homomorphism

$$(3.1) \quad \Pi : C^*(L^1(G, \Gamma)) \longrightarrow p_0 C^*(\overline{G}) p_0,$$

see [15] and [7]. This map is known to be an isomorphism in many cases, such as when  $\overline{G}$  is Hermitian, [7], or when  $L^1(\overline{G})$  is quasi-symmetric in the terminology of [9]. The last property is known to hold for a class of groups containing Hermitian groups and groups with subexponential growth. For a Gelfand pair, the next result gives a necessary and sufficient condition for the canonical surjection to be an isomorphism in terms of properties of  $*$ -spherical functions.

**Theorem 3.1.** *Let  $G$  be a unimodular locally compact group and  $U$  a compact open subgroup such that  $(G, U)$  is a Gelfand pair. The following are equivalent:*

- (a) *All bounded  $*$ -spherical functions for  $\mathcal{H}(G, U)$  are positive definite.*
- (b) *The canonical surjection  $\Pi : C^*(L^1(G, U)) \longrightarrow p_0 C^*(G) p_0$  is an isomorphism.*

**Proof:** We have that  $p_0$  is the self-adjoint projection in  $L^1(G)$  equal to  $\chi_U$ . We aim to use [7, Corollary 6.11(ii)], according to which the Banach  $*$ -algebra  $D := p_0 L^1(G) p_0$  is such that  $C^*(D) = p_0 C^*(G) p_0$  if and only if every  $*$ -representation  $\pi$  of  $D$  on a Hilbert space satisfies

$$(3.2) \quad \pi(\langle f, f \rangle_D) \geq 0 \text{ for all } f \in L^1(G) p_0,$$

where the  $D$ -valued inner product  $\langle \cdot, \cdot \rangle_D$  is defined by  $\langle f_1, f_2 \rangle_D = f_1^* f_2$  for  $f_1, f_2 \in L^1(G) p_0$ .

Let us assume (a). We will establish (3.2), which therefore shows that  $C^*(D) = p_0 C^*(G) p_0$ .

**Claim 1:** The inequality in (3.2) is valid when  $\pi$  runs over the set of characters of  $D$ . To see this, let  $f = \sum_{j=1}^m s_j p_0$  in  $C_c(G) p_0 \subset L^1(G) p_0$ ; then  $f^* = \sum_{j=1}^m p_0 s_j^{-1}$ . Let  $\omega$  be a bounded  $*$ -spherical function, which by assumption is positive definite. Then, using that

$p_0 g p_0 = \frac{1}{L(g)} \chi_{UgU}$  for every  $g \in G$ , it follows that

$$\begin{aligned}
\tau_\omega(\langle f, f \rangle_D) &= \tau_\omega(p_0 \sum_{j,k=1}^m s_j^{-1} s_k p_0) \\
&= \sum_{j,k=1}^m \tau_\omega(p_0 s_j^{-1} s_k p_0) \\
&= \sum_{j,k=1}^m \frac{1}{L(s_j^{-1} s_k)} \int_G \chi_{U s_j^{-1} s_k U}(g) \omega(g^{-1}) d\mu(g) \\
&= \sum_{j,k=1}^m \frac{1}{L(s_j^{-1} s_k)} \int_{U s_j^{-1} s_k U} \omega(s_j^{-1} s_k) d\mu(g) \\
(3.3) \quad &= \sum_{j,k=1}^m \omega(s_j^{-1} s_k).
\end{aligned}$$

Thus  $\tau_\omega(\langle f, f \rangle_D) \geq 0$  for all bounded \*-spherical functions  $\omega$  on  $\mathcal{H}(G, U)$ . By continuity of  $\tau_\omega$ , it follows that  $\tau_\omega(\langle f, f \rangle_D) \geq 0$  for all  $f \in L^1(G)p_0$  and all bounded \*-spherical functions on  $\mathcal{H}(G, U)$ . Since  $D \cong L^1(G, U)$  and every character of  $L^1(G, U)$  is of the form  $\tau_\omega$  for some bounded \*-spherical function  $\omega$ , it follows that

$$\pi(\langle f, f \rangle_D) \geq 0$$

for any character  $\pi$  of  $L^1(G, U)$ . This proves claim 1.

**Claim 2:** If (3.2) is valid when  $\pi$  runs over the set of characters of  $D$ , then it is valid for arbitrary  $\pi$ . To show this, let  $\pi$  be a \*-representation of the commutative, unital Banach \*-algebra  $D$  on a Hilbert space  $H_\pi$  and fix an arbitrary state  $\psi$  of  $B(H_\pi)$ . Then there are pure states  $\tau_j$  of  $D$  and positive real numbers  $\alpha_j$  for  $j \in J$  such that  $\psi \circ \pi = \sum_{j \in J} \alpha_j \tau_j$  and  $\sum_{j \in J} \alpha_j = 1$ , see for example [10, Theorem 9.6.6]. Now

$$(\psi \circ \pi)(\langle f, f \rangle_D) = \sum_{j \in J} \alpha_j \tau_j(\langle f, f \rangle_D),$$

which is non-negative by assumption. Since  $\psi$  was chosen arbitrarily, it follows that  $\pi(\langle f, f \rangle_D) \geq 0$ , proving claim 2 and finishing the proof of one implication.

Conversely, assume (b). Let  $\omega$  be a bounded \*-spherical function on  $\mathcal{H}(G, U)$ . By Propositions 2.2 and 2.3,  $\tau_\omega$  extends to a \*-homomorphism  $L^1(G, U) \rightarrow \mathbb{C}$ . Fix a finite subset  $\{s_1, \dots, s_m\}$  of  $G$  and a finite subset  $\{z_1, \dots, z_m\}$  of complex numbers. Let  $f = \sum_{j=1}^m z_j s_j p_0 \in L^1(G)p_0$ . Since  $\tau_\omega$  satisfies (3.2), a computation similar to the one leading to equation (3.3) shows that  $\sum_{j,k=1}^m \omega(s_j^{-1} s_k) \overline{z_j} z_k = \tau_\omega(\langle f, f \rangle_D) \geq 0$ , as needed in (a).  $\square$

The Hecke algebra  $\mathcal{H}(G, \Gamma)$  associated to a Hecke pair  $(G, \Gamma)$  need not admit a universal  $C^*$ -completion, [6]. When it does, the universal  $C^*$ -completion  $C^*(G, \Gamma)$  of  $\mathcal{H}(G, \Gamma)$  admits a natural surjection onto  $C^*(L^1(G, \Gamma))$ . As pointed out in [7, Questions 6.16(ii)], this map is an isomorphism for all known classes such that  $C^*(G, \Gamma)$  exists, although a general explanation of why this must be the case is missing. The next result gives a necessary and sufficient condition for existence of a universal  $C^*$ -completion of a Gelfand pair. As in all other similar cases, the natural surjection then becomes injective.

**Theorem 3.2.** *Let  $G$  be a unimodular locally compact group and  $U$  a compact open subgroup such that  $(G, U)$  is a Gelfand pair. The following are equivalent:*

- (a) All  $*$ -spherical functions for  $\mathcal{H}(G, U)$  are bounded.  
(b) The universal  $C^*$ -completion  $C^*(G, U)$  exists and the canonical surjection  $C^*(G, U) \longrightarrow C^*(L^1(G, U))$  is an isomorphism.

**Proof:** Let us assume (a) is true. Since  $\mathcal{H}(G, U)$  is abelian, in order to prove that  $C^*(G, U)$  exists it suffices to show that  $\sup_{\phi} |\phi(f)| < \infty$  for every  $f \in \mathcal{H}(G, U)$ , where the supremum runs over the set of  $*$ -homomorphisms  $\phi : \mathcal{H}(G, U) \rightarrow \mathbb{C}$ . We know that each  $*$ -homomorphism  $\mathcal{H}(G, U) \rightarrow \mathbb{C}$  is of the form  $\tau_{\omega}$  for some  $*$ -spherical function  $\omega$ . By (a) every  $*$ -spherical function is bounded, and therefore  $\tau_{\omega}$  extends to a character of  $L^1(G, U)$ . Hence

$$|\tau_{\omega}(f)| \leq \|f\|_{L^1(G, U)},$$

from which it follows immediately that  $\sup_{\phi} |\phi(f)| \leq \|f\|_{L^1(G, U)} < \infty$ . We conclude that  $C^*(G, U)$  exists. Moreover, since every  $*$ -homomorphism  $\mathcal{H}(G, U) \rightarrow \mathbb{C}$  extends to a character of  $L^1(G, U)$  it follows that the canonical surjection  $C^*(G, U) \longrightarrow C^*(L^1(G, U))$  is an isomorphism.

Let us now assume (b). Let  $\omega$  be a  $*$ -spherical function. As we know,  $\tau_{\omega}$  is then a  $*$ -homomorphism  $\mathcal{H}(G, U) \rightarrow \mathbb{C}$ . By (b) we must have

$$|\tau_{\omega}(f)| \leq \|f\|_{C^*(G, U)} = \|f\|_{C^*(L^1(G, U))} \leq \|f\|_{L^1(G, U)}.$$

Hence,  $\tau_{\omega}$  extends to a character of  $L^1(G, U)$ . Thus,  $\omega$  must be bounded.  $\square$

An immediate consequence of Theorems 3.1 and 3.2 is the following:

**Corollary 3.3.** *Let  $G$  be a unimodular locally compact group and  $U$  a compact open subgroup such that  $(G, U)$  is a Gelfand pair. The following are equivalent:*

- (a) All  $*$ -spherical functions for  $\mathcal{H}(G, U)$  are bounded and positive definite.  
(b) the universal  $C^*$ -completion  $C^*(G, U)$  exists and the natural maps are isomorphisms:

$$C^*(G, U) \xrightarrow{\cong} C^*(L^1(G, U)) \xrightarrow{\cong} p_0 C^*(G) p_0.$$

#### 4. SPHERICAL FUNCTIONS FOR $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$

The spherical functions for the pair  $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$  were completely characterised by Satake, see [12] and in particular [13, §7.3]. In this section we introduce the following terminology, in accordance with [13, Chapter III]. Thus we let

- $G := SL_n(\mathbb{Q}_p)$ ,
- $U := SL_n(\mathbb{Z}_p)$ ,
- $H := \{h \in G : h \text{ is diagonal}\}$ ,
- $N := \{g \in G : g \text{ is upper-triangular and has 1's on the diagonal}\}$ ,
- $H^u := \{h \in G : h = \text{diag}(h_1, \dots, h_n), \text{ with } h_k \in \mathbb{Z}_p^* \text{ for } k = 1, \dots, n\}$ ,
- $X(H)$ , the module of  $\mathbb{Q}_p$ -morphisms of  $H$  into  $\mathbb{Q}_p^*$ ,
- $X(H) \otimes \mathbb{C} \cong \mathbb{C}^n / \{(s, \dots, s) : s \in \mathbb{C}\}$ ,
- $M \cong \{(m_1, \dots, m_n) \in \mathbb{Z}^n : \sum_{k=1}^n m_k = 0\}$ ,
- $\widehat{M} := \{s \in X(H) \otimes \mathbb{C} : \mathbf{m} \cdot s \in \mathbb{Z} \text{ for all } \mathbf{m} \in M\}$ ,
- $W = \mathfrak{S}_n$ , the symmetric group on  $n$  letters, which acts on  $M$  by permuting the coordinates.
- For  $\mathbf{m} = (m_1, \dots, m_n) \in M$  we use the notation  $\pi^{\mathbf{m}} := \begin{pmatrix} p^{m_1} & & \\ & \ddots & \\ & & p^{m_n} \end{pmatrix}$ .



Under the identifications for  $X(H) \otimes \mathbb{C}$  and  $M$  above, if  $\mathbf{m} = (m_1, \dots, m_n) \in M$  and  $[\mathbf{s}] \in X(H) \otimes \mathbb{C}$  with  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ , then

$$\mathbf{m} \cdot \mathbf{s} = \sum_{k=1}^n m_k s_k.$$

Following [13, §5.4], if  $\alpha : H \rightarrow \mathbb{C}^*$  is a quasi-character (i.e. a continuous homomorphism of  $H$  into  $\mathbb{C}^*$  with respect to the  $p$ -adic topology) satisfying  $\alpha(H^u) = 1$ , then  $\alpha$  is uniquely determined by the values of  $\alpha(\pi^{\mathbf{m}})$  with  $\mathbf{m} \in M$ , and there exists  $[\mathbf{s}] \in X(H) \otimes \mathbb{C}$  such that

$$(4.1) \quad \alpha(\pi^{\mathbf{m}}) = p^{-\mathbf{m} \cdot \mathbf{s}}.$$

Whenever  $\alpha(H^u) = 1$  and (4.1) holds, we use Satake's notation and write  $\alpha \leftrightarrow \mathbf{s}$ .

Let  $\delta$  be the positive quasi-character introduced in [13, §4.1]. For  $\mathbf{m} \in M$  we have that

$$(4.2) \quad \delta(\pi^{\mathbf{m}}) = p^{\sum_{k=1}^n n_k + 1 - 2k}.$$

Moreover, if  $\delta^{\frac{1}{2}}\alpha \leftrightarrow \mathbf{s}$  we have

$$(4.3) \quad \alpha(\pi^{\mathbf{m}}) = p^{-\sum_{k=1}^n m_k(s_k + k - \frac{n+1}{2})}.$$

By [13, §5.3 and §5.4], whenever  $\delta^{\frac{1}{2}}\alpha \leftrightarrow \mathbf{s}$  we can define a spherical function  $\omega_{\mathbf{s}}$  in the following way:

$$(4.4) \quad \omega_{\mathbf{s}}(g) := \int_U \psi_{\alpha}(g^{-1}u) du,$$

where  $\psi_{\alpha}(uhn) := \alpha(h)$  for  $u \in U$ ,  $h \in H$  and  $n \in N$ . Then the following properties hold:

$$(4.5) \quad \omega_{-\mathbf{s}}(g) = \omega_{\mathbf{s}}(g^{-1})$$

$$(4.6) \quad \omega_{w\mathbf{s}} = \omega_{\mathbf{s}},$$

for all  $w \in W$ . Moreover  $\omega_{\mathbf{s}} = \omega_{\mathbf{s}'}$  if and only if  $\mathbf{s}$  and  $\mathbf{s}'$  are equivalent with respect to the action of  $W \cdot \left(\frac{2\pi i}{\log p} \widehat{M}\right)$  (here  $i \in \mathbb{C}$  is the imaginary unit).

Let  $\overline{\mathbf{s}} := (\overline{s_1}, \dots, \overline{s_n})$  be the element obtained from  $\mathbf{s}$  by taking the complex conjugate of all of its entries.

**Proposition 4.1.** *If  $\mathbf{s} = -w\overline{\mathbf{s}}$  for some  $w \in W$ , then  $\omega_{\mathbf{s}}$  is  $*$ -spherical.*

**Proof:** By (4.5) and (4.6) we have  $\omega_{\mathbf{s}}(g^{-1}) = \omega_{-\mathbf{s}}(g) = \omega_{w\overline{\mathbf{s}}}(g) = \omega_{\overline{\mathbf{s}}}(g)$ .

For any  $[\mathbf{r}] \in X(H) \otimes \mathbb{C}$  let  $\alpha_{\mathbf{r}} : H \rightarrow \mathbb{C}^*$  be the quasi-character determined by

$$\alpha_{\mathbf{r}}(\pi^{\mathbf{m}}) = p^{-\sum_{k=1}^n m_k(r_k + k - \frac{n+1}{2})}.$$

By (4.3) one has  $\delta^{\frac{1}{2}}\alpha_{\mathbf{r}} \leftrightarrow \mathbf{r}$ . For the element  $\mathbf{s}$  we then have

$$\begin{aligned} \alpha_{\overline{\mathbf{s}}}(\pi^{\mathbf{m}}) &= p^{-\sum_{k=1}^n m_k(\overline{s_k} + k - \frac{n+1}{2})} \\ &= p^{-\sum_{k=1}^n m_k(s_k + k - \frac{n+1}{2})} \\ &= \overline{\left(p^{-\sum_{k=1}^n m_k(s_k + k - \frac{n+1}{2})}\right)} \\ &= \overline{\alpha_{\mathbf{s}}(\pi^{\mathbf{m}})}, \end{aligned}$$

and therefore  $\alpha_{\overline{\mathbf{s}}}(h) = \overline{\alpha_{\mathbf{s}}(h)}$  for all  $h \in H$ . Moreover, it is then also true that  $\psi_{\alpha_{\overline{\mathbf{s}}}} = \overline{\psi_{\alpha_{\mathbf{s}}}}$ . Hence, we have

$$\omega_{\overline{\mathbf{s}}}(g) = \int_U \psi_{\alpha_{\overline{\mathbf{s}}}}(g^{-1}u) du = \int_U \overline{\psi_{\alpha_{\mathbf{s}}}(g^{-1}u)} du = \overline{\omega_{\mathbf{s}}(g)}.$$

Thus we conclude that  $\omega_{\mathbf{s}}(g^{-1}) = \overline{\omega_{\mathbf{s}}(g)}$ .  $\square$

We claim that the trivial spherical function  $\omega_{\mathbf{t}} = 1$  is attained at (the orbit of) the point  $\mathbf{t} = (t_1, \dots, t_n)$  where  $t_k = \frac{n+1}{2} - k$  for  $k = 1, \dots, n$ . This can be seen from (4.3) since if  $\alpha$  is such that  $\delta^{\frac{1}{2}}\alpha \leftrightarrow \mathbf{t}$ , then it is clear that  $\alpha = 1$ , and therefore  $\psi_{\alpha} = 1$  and consequently  $\omega_{\mathbf{t}} = 1$ .

**Theorem 4.2.** *Suppose  $n \geq 3$ . Let  $\mathbf{r} := (1, 0, \dots, 0, 1)$  in  $\mathbb{C}^n$  and define a sequence  $\mathbf{s}(j) := \mathbf{t} + \frac{j}{j}\mathbf{r}$  for  $j \in \mathbb{N}$ . Then  $\{\omega_{\mathbf{s}(j)}\}_{j \in \mathbb{N}}$  is a sequence of mutually different, bounded  $*$ -spherical functions. Moreover, we have that  $\omega_{\mathbf{s}(j)} \rightarrow \omega_{\mathbf{t}}$  as  $j \rightarrow \infty$  uniformly on compact sets.*

**Proof:** For each  $j \in \mathbb{N}$  let  $\alpha_j$  be such that  $\delta^{\frac{1}{2}}\alpha_j \leftrightarrow \mathbf{s}(j)$ . For fixed  $j$  and any  $\mathbf{m} \in M$  we have

$$\begin{aligned} \alpha_j(\pi^{\mathbf{m}}) &= p^{-\sum_{k=1}^n m_k(s(j)_k + k - \frac{n+1}{2})} \\ (4.7) \quad &= p^{i\left(-\frac{m_1}{j} - \frac{m_n}{j}\right)}. \end{aligned}$$

Hence  $|\alpha_j(\pi^{\mathbf{m}})| = 1$ . It follows that  $|\alpha_j(h)| = 1$  for all  $h \in H$ , and therefore  $|\psi_{\alpha_j}(g)| = 1$  for all  $g \in G$ . Thus we have that  $\omega_{\mathbf{s}(j)}$  is a bounded spherical function, because

$$|\omega_{\mathbf{s}(j)}(g)| \leq \int_U |\psi_{\alpha_j}(g^{-1}u)| du = 1.$$

Let  $w_0 \in W$  be the permutation that takes the  $k$ -th coordinate to the  $(n-k+1)$ -th coordinate for each  $k = 1, \dots, n$ . Since  $t_{n-k+1} = -t_k$  for every  $k = 1, \dots, n$ , we have that

$$-w_0\overline{\mathbf{s}(j)} = -w_0(\mathbf{t} - \frac{j}{j}\mathbf{r}) = \mathbf{t} + \frac{j}{j}\mathbf{r} = \mathbf{s}(j),$$

so that  $\omega_{\mathbf{s}(j)}$  is  $*$ -spherical, by Proposition 4.1.

We show next that any two functions  $\omega_{\mathbf{s}(j)}$  and  $\omega_{\mathbf{s}(j')}$  are different whenever  $j \neq j'$ . Fix therefore  $j \neq j'$  and suppose that  $\omega_{\mathbf{s}(j)} = \omega_{\mathbf{s}(j')}$ . Then we know that  $\mathbf{s}(j)$  and  $\mathbf{s}(j')$  are equivalent with respect to the action of  $W(\frac{2\pi i}{\log p}\widehat{M})$ , i.e. there exists  $w \in W$  such that

$$\mathbf{s}(j') - w\mathbf{s}(j) \in \frac{2\pi i}{\log p}\widehat{M}.$$

We will now show that  $\mathbf{s}(j') - w\mathbf{s}(j)$  cannot be an element of  $\frac{2\pi i}{\log p}\widehat{M}$ , thus arriving at a contradiction. Let us first introduce the following notation: for  $k \in \{1, \dots, n\}$ , the element  $\mathbf{m}_{(1,k)} \in M$  is the element which has 1 in the 1-st coordinate,  $-1$  in the  $k$ -th coordinate and is zero elsewhere. We denote by  $\text{Im}[\mathbf{s}(j)]_m$  the imaginary part of the  $m$ -th coordinate of  $\mathbf{s}(j)$ , for  $m \in \{1, \dots, n\}$ .

Fix any coordinate  $k$  with  $1 \neq k \neq n$ , which we may because  $n \geq 3$ . We have that

$$(4.8) \quad (\mathbf{s}(j') - w\mathbf{s}(j)) \cdot \mathbf{m}_{(1,k)} = x + i\left(\frac{1}{j'} - \text{Im}[w\mathbf{s}(j)]_1 + \text{Im}[w\mathbf{s}(j)]_k\right),$$

for some real number  $x \in \mathbb{R}$ . It is clear that the imaginary part of (4.8) is a number in  $\mathbb{Q}$ . Moreover, it cannot be zero since  $\text{Im}[w\mathbf{s}(j)]_1$  and  $\text{Im}[w\mathbf{s}(j)]_k$  can only take the values 0 or  $\frac{1}{j}$ . Therefore, by looking at the imaginary part of (4.8), it follows that  $\mathbf{s}(j') - w\mathbf{s}(j)$  cannot be an element of  $\frac{2\pi i}{\log p}\widehat{M}$ . This proves that  $\omega_{\mathbf{s}(j)}$  and  $\omega_{\mathbf{s}(j')}$  are different, as claimed.

It remains to show that  $\omega_{\mathbf{s}(j)} \rightarrow \omega_{\mathbf{t}}$  uniformly on compact sets. Since spherical functions are constant on the compact open sets  $UgU$ , with  $g \in G$ , it is enough to prove pointwise convergence, i.e.  $\omega_{\mathbf{s}(j)}(g) \rightarrow \omega_{\mathbf{t}}(g)$  for any  $g \in G$ .

Fix therefore  $g \in G$ . We have that  $g^{-1}U \subseteq Ug^{-1}U$ , and furthermore we can write  $Ug^{-1}U$  as a finite union of right cosets

$$Ug^{-1}U = \bigcup_{l=1}^L Ug^{-1}u_l,$$

where  $u_l \in U$  for all  $l = 1, \dots, L$ . For each  $l$  we can write  $g^{-1}u_l = u'_l h_l n_l$ , where  $u'_l \in U$ ,  $h_l \in H$  and  $n_l \in N$ . Let  $\mathbf{m}(l) \in M$  be such that  $h_l \in \pi^{\mathbf{m}(l)} H^u$ . Thus we have  $\psi_{\alpha_j}(Ug^{-1}u_l) = \alpha_j(h_l) = \alpha_j(\pi^{\mathbf{m}(l)})$ .

Using the partition  $U = \bigcup_{l=1}^L [U \cap gUg^{-1}]u_l$  and invoking (4.7) in the last step gives that

$$\begin{aligned} |\omega_{\mathbf{s}(j)}(g) - \omega_{\mathbf{t}}(g)| &= \left| \int_U \psi_{\alpha_j}(g^{-1}u) du - 1 \right| \\ &\leq \int_U |\psi_{\alpha_j}(g^{-1}u) - 1| du \\ &= \sum_{l=1}^L \int_{[U \cap gUg^{-1}]u_l} |\psi_{\alpha_j}(g^{-1}u) - 1| du \\ &= \sum_{l=1}^L \int_{[U \cap gUg^{-1}]u_l} |\alpha_j(\pi^{\mathbf{m}(l)}) - 1| du \\ &\leq \sum_{l=1}^L |\alpha_j(\pi^{\mathbf{m}(l)}) - 1| \\ &= \sum_{l=1}^L \left| p^{-\frac{i(m(l)_1 + m(l)_n)}{j}} - 1 \right|. \end{aligned}$$

The last expression goes to zero as  $j \rightarrow \infty$ . Thus the proof of the theorem is complete.  $\square$

## 5. PROPERTY (T) AND THE CANONICAL SURJECTION FOR $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$

Let  $A$  be a  $*$ -algebra. We will denote by  $\text{Prim}(A)$  the *primitive ideal space* of  $A$ , i.e. the set of kernels of topologically irreducible  $*$ -representations of  $A$ , with the hull-kernel topology. Moreover, we will denote by  $\widehat{A}$  the *dual space* of  $A$ , which is the set of unitary equivalence classes of irreducible  $*$ -representations of  $A$ . There is a canonical map  $\widehat{A} \rightarrow \text{Prim}(A)$  defined by  $[\pi] \mapsto \text{Ker}(\pi)$ , and the topology of  $\widehat{A}$  is the topology pulled back from  $\text{Prim}(A)$  through this map.

Property (T) for a topological group is known to have several equivalent formulations. For our purposes two of these characterisations are crucial: one of them because it captures a property (T) for Hecke pairs, as introduced by Tzanev in [16], see Appendix A and in particular Theorem A.7. The other characterisation has our interest because it *fails* to pass to Hecke pairs. As we shall now show, this failure is intimately related to the question of whether the canonical surjection from (3.1) is injective.

**Proposition 5.1.** *Let  $(G, \Gamma)$  be a Hecke pair with Schlichting completion  $(\overline{G}, \overline{\Gamma})$ . Suppose that  $\overline{G}$  has property (T) or, equivalently, by Theorem A.7 the pair  $(G, \Gamma)$  has property (T) for Hecke pairs. Then the trivial representation of  $p_0 C^*(\overline{G}) p_0$  is an isolated point of  $(p_0 C^*(\overline{G}) p_0)^\wedge$  with the hull-kernel topology.*

**Proof:** Since  $\overline{G}$  has property (T), the trivial representation  $\mathbf{1}_{\overline{G}}$  of  $\overline{G}$  is isolated in  $\widehat{\overline{G}}$  with the Fell topology. Under the standard identifications, the Fell topology on  $\widehat{\overline{G}}$  coincides with the hull-kernel topology on  $C^*(\overline{G})^\wedge$  (see [4, §18] and comment on page 438 of [BdHV]), so that the trivial representation of  $C^*(\overline{G})$  is an isolated point of  $(C^*(\overline{G}))^\wedge$ . Moreover, by [11, Proposition A.27 (b)], we see that  $\mathbf{1}_{\overline{G}}$ , restricted to the ideal  $\overline{C^*(\overline{G})p_0C^*(\overline{G})}$ , is an isolated point of  $(\overline{C^*(\overline{G})p_0C^*(\overline{G})})^\wedge$ . By the Morita equivalence between  $\overline{C^*(\overline{G})p_0C^*(\overline{G})}$  and  $p_0C^*(\overline{G})p_0$ , together with [11, Corollary 3.33 (a)], we see that the trivial representation of  $p_0C^*(\overline{G})p_0$  is an isolated point of  $(p_0C^*(\overline{G})p_0)^\wedge$ .  $\square$

The above proposition shows that  $p_0C^*(\overline{G})p_0$  still captures the characterisation of property (T) in terms of the isolation of the trivial representation. However, as the next result shows, this characterization of property (T) fails in general to pass to  $C^*(L^1(\overline{G}, \overline{\Gamma}))$ .

**Theorem 5.2.** *Let  $G = SL_n(\mathbb{Q}_p)$  and  $U = SL_n(\mathbb{Z}_p)$  for some  $n \geq 3$ . The trivial representation of  $C^*(L^1(G, U))$  is not an isolated point of the space  $(C^*(L^1(G, U)))^\wedge$  endowed with its hull-kernel topology.*

**Proof:** By Theorem 4.2 there is a sequence of (mutually different) bounded  $*$ -spherical functions  $\{\omega_{\mathbf{s}(j)}\}_{j \in \mathbb{N}}$  that converges uniformly on compact sets to the trivial spherical function  $\omega_{\mathbf{t}}$ .

By Propositions 4.1 and 2.2, each  $\tau_{\omega_{\mathbf{s}(j)}}$  is a  $*$ -homomorphism of  $L^1(G, U)$  onto  $\mathbb{C}$ . Moreover, by Proposition 2.5, the sequence  $\{\tau_{\omega_{\mathbf{s}(j)}}\}_{j \in \mathbb{N}}$  converges to the trivial representation  $\tau_{\omega_{\mathbf{t}}}$  of  $L^1(G, U)$  in the weak\*-topology.

Let us denote by  $L^1(G, U)^\wedge$  the space of non-trivial  $*$ -homomorphisms of  $L^1(G, U)$  onto  $\mathbb{C}$  endowed with the weak\*-topology, which is the same as the state space and the pure state space of  $L^1(G, U)$ , since this Banach  $*$ -algebra is abelian. Since the  $*$ -homomorphisms  $\tau_{\omega_{\mathbf{s}(j)}}$  are mutually different, the trivial representation of  $L^1(G, U)$  is not an isolated point of  $L^1(G, U)^\wedge$ .

By [10, Theorem 10.1.12 (b)] the spaces  $(C^*(L^1(G, U)))^\wedge$  and  $L^1(G, U)^\wedge$  are homeomorphic. Hence, the trivial representation of  $C^*(L^1(G, U))$  is not an isolated point of  $(C^*(L^1(G, U)))^\wedge$  with the weak\*-topology. Since  $C^*(L^1(G, U))$  is abelian, the weak\*-topology and the hull-kernel topologies coincide (by, for example, [11, Appendix A]), and this proves our claim.  $\square$

For  $n \geq 3$  it is known that  $SL_n(\mathbb{Q}_p)$  has property (T), see for example [1]. Hence by Proposition A.6,  $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$  has property (T) for Hecke pairs. Further, by Proposition 5.1 we have that the trivial representation of  $p_0C^*(G)p_0$  is isolated in  $(p_0C^*(G)p_0)^\wedge$ . This fact and Theorem 5.2 lead to the following corollary:

**Corollary 5.3.** *Let  $(G, U) = (SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$  with  $n \geq 3$ . The two  $C^*$ -completions  $C^*(L^1(G, U))$  and  $p_0C^*(G)p_0$  of the Hecke algebra  $\mathcal{H}(G, U)$  do not coincide, i.e. the canonical surjection*

$$C^*(L^1(G, U)) \longrightarrow p_0C^*(G)p_0$$

*is not injective.*

At the same time, we see that isolation of the trivial representation in its natural topologies does not pass from  $SL_n(\mathbb{Q}_p)$  to the Hecke pair  $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$  for  $n \geq 3$ .

## APPENDIX A. TZANEV'S PROPERTY T AND RELATION TO THE SCHLICHTING COMPLETION

In [16, Chapitre 1], Tzanev defined a property (T) for a Hecke pair  $(G, \Gamma)$ , and claimed without proof that a Hecke pair  $(G, \Gamma)$  has property (T) if and only if the Schlichting completion  $\overline{G}$  has property (T) in the ordinary sense for groups. He further claimed that all other characterisations of property (T) translate to Hecke pairs. We have already seen that the last assertion fails for  $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$  when  $n \geq 3$ .

Tzanev's first claim is true, but in filling out the details of the proof we have found a more involved argument than expected. We include the proof in this appendix, in the hope that the result could be useful elsewhere.

Recall the definition of property (T) for locally compact groups, [1]. Let  $\mathcal{G}$  be a locally compact group and  $\pi : \mathcal{G} \rightarrow \mathcal{U}(H)$  a unitary representation on a Hilbert space  $H$ . Given  $\varepsilon > 0$  and  $\mathcal{Q}$  a compact subset of  $\mathcal{G}$ , a vector  $\xi \in H$  with  $\|\xi\| = 1$  is  $(\varepsilon, \mathcal{Q})$ -invariant if  $\|\pi(g)\xi - \xi\| < \varepsilon$  for all  $g \in \mathcal{Q}$ . The representation  $\pi$  has almost  $\mathcal{G}$ -invariant vectors if it admits an  $(\varepsilon, \mathcal{Q})$ -invariant vector for any  $\varepsilon > 0$  and  $\mathcal{Q}$  compact subset of  $\mathcal{G}$ . The group  $\mathcal{G}$  has property (T) of Kazhdan if every unitary representation of  $\mathcal{G}$  having almost  $\mathcal{G}$ -invariant vectors has a nontrivial  $\mathcal{G}$ -invariant vector.

By analogy with the above, Tzanev introduced a notion of property (T) for a Hecke pair  $(G, \Gamma)$ . We shall phrase the definition in terms of a topological Hecke pair  $(\mathcal{G}, \mathcal{H})$ . In fact, the definition makes sense only assuming that  $\mathcal{H}$  is a closed subgroup of  $\mathcal{G}$ .

**Definition A.1.** Let  $\mathcal{G}$  be a locally compact group,  $\mathcal{H}$  a closed subgroup and  $\pi : \mathcal{G} \rightarrow \mathcal{U}(H)$  a unitary representation of  $\mathcal{G}$  on a Hilbert space  $H$ . Given  $\varepsilon > 0$  and  $\mathcal{Q}$  a compact subset of  $\mathcal{G}/\mathcal{H}$ , a vector  $\xi$  in  $H$  with  $\|\xi\| = 1$  is  $(\varepsilon, \mathcal{Q})$ -invariant if  $\xi$  is  $\mathcal{H}$ -invariant and  $\|\pi(g)\xi - \xi\| < \varepsilon$  for all  $[g] \in \mathcal{Q}$ .

The representation  $\pi$  has  *$\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors* if for every compact subset  $\mathcal{Q} \subseteq \mathcal{G}/\mathcal{H}$  and every  $\varepsilon > 0$  there exists a  $(\varepsilon, \mathcal{Q})$ -invariant vector.

The pair  $(\mathcal{G}, \mathcal{H})$  has *property (T)* if every unitary representation of  $\mathcal{G}$  having  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors has a nontrivial  $\mathcal{G}$ -invariant vector.

In the case of a discrete Hecke pair  $(G, \Gamma)$ , the above recovers Tzanev's definition upon replacing  $\mathcal{Q}$  with a finite subset  $K/\Gamma$  of  $G/\Gamma$  for  $K$  a subset of  $G$ .

**Remark A.2.** The above notion of property (T) is not the same as the relative property (T) for pairs. On one hand,  $(\mathcal{G}, \mathcal{G})$  always has property (T) in the sense of Definition A.1, but only has the relative property (T) when  $\mathcal{G}$  has property (T). On the other hand,  $(\mathcal{G}, \{e\})$  always has the relative property (T), but only has property (T) in the sense of Definition A.1 when  $\mathcal{G}$  has property (T).

**Proposition A.3.** Let  $\mathcal{G}$  be a topological group and  $\mathcal{N} \subseteq \mathcal{H} \subseteq \mathcal{G}$  closed subgroups, with  $\mathcal{N}$  a normal subgroup of  $\mathcal{G}$ . The pair  $(\mathcal{G}, \mathcal{H})$  has property (T) if and only if  $(\mathcal{G}/\mathcal{N}, \mathcal{H}/\mathcal{N})$  has property (T).

In particular, for a topological group  $\mathcal{G}$  and a closed normal subgroup  $\mathcal{N} \trianglelefteq \mathcal{G}$ , we have that  $(\mathcal{G}, \mathcal{N})$  has property (T) if and only if  $\mathcal{G}/\mathcal{N}$  has property (T).

**Proof:** ( $\implies$ ) Suppose  $(\mathcal{G}, \mathcal{H})$  has property (T). Let  $\pi : \mathcal{G}/\mathcal{N} \rightarrow \mathcal{U}(H)$  be a unitary representation that has  $\mathcal{H}/\mathcal{N}$ -invariant almost  $\mathcal{G}/\mathcal{N}$ -invariant vectors. We want to show that  $\pi$  has a nontrivial  $\mathcal{G}/\mathcal{N}$ -invariant vector. Let  $\tilde{\pi} := \pi \circ q$  be the lifting of  $\pi$  to  $\mathcal{G}$  through the quotient map  $q : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{N}$ . Let  $\mathcal{Q} \subseteq \mathcal{G}/\mathcal{H}$  be a compact set and  $\varepsilon > 0$ . Under the canonical homeomorphism between  $\mathcal{G}/\mathcal{H}$  and  $(\mathcal{G}/\mathcal{N})/(\mathcal{H}/\mathcal{N})$  we can naturally identify  $\mathcal{Q}$

with a compact subset  $\mathcal{Q}' \subseteq (\mathcal{G}/\mathcal{N})/(\mathcal{H}/\mathcal{N})$ . Thus, there exists a  $\mathcal{H}/\mathcal{N}$ -invariant vector  $\xi \in H$  of norm one such that

$$(A.1) \quad \|\pi(g\mathcal{N})\xi - \xi\| < \varepsilon$$

for every  $g\mathcal{N}$  such that  $[g\mathcal{N}] \in \mathcal{Q}' \subseteq (\mathcal{G}/\mathcal{N})/(\mathcal{H}/\mathcal{N})$ . The following diagram of canonical maps commutes:

$$(A.2) \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad} & \mathcal{G}/\mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{G}/\mathcal{H} & \xrightarrow{\cong} & \frac{\mathcal{G}/\mathcal{N}}{\mathcal{H}/\mathcal{N}} \end{array}$$

Therefore the set of elements  $g \in \mathcal{G}$  such that  $[g\mathcal{N}] \in \mathcal{Q}'$  is the same as the set of elements  $g \in \mathcal{G}$  such that  $g\mathcal{H} \in \mathcal{Q}$ . Thus, condition (A.1) simply says that  $\|\tilde{\pi}(g)\xi - \xi\| < \varepsilon$  for all  $g \in \mathcal{G}$  such that  $g\mathcal{H} \in \mathcal{Q}$ . It is moreover clear that  $\xi$  is  $\mathcal{H}$ -invariant for  $\tilde{\pi}$ . Hence,  $\tilde{\pi}$  has  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors. From property (T) it then follows that  $\tilde{\pi}$  has a true  $\mathcal{G}$ -invariant vector, say  $\xi_0 \in H$ . It is clear that  $\xi_0$  is then a  $\mathcal{G}/\mathcal{N}$ -invariant vector for  $\pi$ .

( $\Leftarrow$ ) Suppose  $(\mathcal{G}/\mathcal{N}, \mathcal{H}/\mathcal{N})$  has property (T). Let  $\pi : \mathcal{G} \rightarrow \mathcal{U}(H)$  be a unitary representation that has  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors. We want to show that  $\pi$  has a true  $\mathcal{G}$ -invariant vector. Let us consider the subspace  $H^{\mathcal{N}}$  of  $\mathcal{N}$ -invariant vectors, i.e.  $H^{\mathcal{N}} := \{\eta \in H : \pi(h)\eta = \eta, \forall h \in \mathcal{N}\}$ , which is nontrivial because  $\pi$  has  $\mathcal{H}$ -invariant (hence also  $\mathcal{N}$ -invariant) vectors. Consider now the unitary representation  $\sigma : \mathcal{G}/\mathcal{N} \rightarrow \mathcal{U}(H^{\mathcal{N}})$  defined by  $\sigma([g])\eta := \pi(g)\eta$ . Note that this is a representation on  $H^{\mathcal{N}}$ , i.e.  $\pi(g)\eta \in H^{\mathcal{N}}$  whenever  $\eta \in H^{\mathcal{N}}$ , because normality of  $\mathcal{N}$  in  $\mathcal{G}$  yields that

$$\pi(h)\pi(g)\eta = \pi(g)\pi(g^{-1}hg)\eta = \pi(g)\eta.$$

We claim that  $\sigma$  has  $\mathcal{H}/\mathcal{N}$ -invariant almost  $\mathcal{G}/\mathcal{N}$ -invariant vectors. Let  $\mathcal{Q} \subseteq (\mathcal{G}/\mathcal{N})/(\mathcal{H}/\mathcal{N})$  be a compact set and  $\varepsilon > 0$ . Under the homeomorphism between  $\mathcal{G}/\mathcal{H}$  and  $(\mathcal{G}/\mathcal{N})/(\mathcal{H}/\mathcal{N})$  we can identify  $\mathcal{Q}$  with a compact subset  $\mathcal{Q}' \subseteq \mathcal{G}/\mathcal{H}$ . Thus, there exists an  $\mathcal{H}$ -invariant vector  $\xi \in H$  such that

$$(A.3) \quad \|\pi(g)\xi - \xi\| < \varepsilon$$

for all  $g \in \mathcal{G}$  such that  $g\mathcal{H} \in \mathcal{Q}'$ . Being  $\mathcal{H}$ -invariant (hence  $\mathcal{N}$ -invariant) for  $\pi$  means that  $\xi \in H^{\mathcal{N}}$  and moreover it implies that  $\xi$  is  $\mathcal{H}/\mathcal{N}$ -invariant for  $\sigma$ . Condition (A.3) simply says that  $\|\sigma([g])\xi - \xi\| < \varepsilon$  for all  $g \in \mathcal{G}$  such that  $g\mathcal{H} \in \mathcal{Q}'$ . By commutativity of the diagram (A.2), it follows that  $\|\sigma([g])\xi - \xi\| < \varepsilon$  for all  $g \in \mathcal{G}$  such that  $[g\mathcal{N}] \in \mathcal{Q}$ . Thus,  $\sigma$  has  $\mathcal{H}/\mathcal{N}$ -invariant almost  $\mathcal{G}/\mathcal{N}$ -invariant vectors and therefore, by property (T) for the pair  $(\mathcal{G}/\mathcal{N}, \mathcal{H}/\mathcal{N})$ , it must have a nontrivial  $\mathcal{G}/\mathcal{N}$ -invariant vector  $\xi_0 \in H^{\mathcal{N}}$ . It is clear that  $\xi_0$  is then a  $\mathcal{G}$ -invariant vector for  $\pi$ .

The second claim of this proposition, that  $(\mathcal{G}, \mathcal{N})$  has property (T) if and only if  $\mathcal{G}/\mathcal{N}$  has Property (T), follows directly from the first claim of the proposition by taking  $\mathcal{H} = \mathcal{N}$ .  $\square$

The following result generalizes the known fact that Property (T) passes to quotients.

**Proposition A.4.** *Let  $\mathcal{G}$  be a topological group and  $\mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{G}$  closed subgroups. If  $(\mathcal{G}, \mathcal{H})$  has property (T), then  $(\mathcal{G}, \mathcal{K})$  has property (T).*

*In particular, if  $\mathcal{G}$  has property (T), then  $(\mathcal{G}, \mathcal{H})$  has property (T) for any closed subgroup  $\mathcal{H} \subseteq \mathcal{G}$ .*

We shall need the following result, whose routine proof we omit.

**Lemma A.5.** *Let  $\mathcal{G}$  be a topological group and  $\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{G}$  closed subgroups. The natural map  $\psi : \mathcal{G}/\mathcal{K} \rightarrow \mathcal{G}/\mathcal{H}$  defined by  $\psi(g\mathcal{K}) := g\mathcal{H}$  is continuous.*

**Proof of Proposition A.4:** We assume first that  $(\mathcal{G}, \mathcal{K})$  has property (T). Let  $\pi : \mathcal{G} \rightarrow \mathcal{U}(H)$  be a unitary representation with  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors. We must show that  $\pi$  has a true nonzero  $\mathcal{G}$ -invariant vector. That will follow immediately from property (T) of  $(\mathcal{G}, \mathcal{K})$  if we prove that  $\pi$  has  $\mathcal{K}$ -invariant almost  $\mathcal{G}$ -invariant vectors, which we will now show.

Let  $\mathcal{Q} \subseteq \mathcal{G}/\mathcal{K}$  be any compact set and let  $\varepsilon > 0$ . Then  $\psi(\mathcal{Q})$  is a compact set of  $\mathcal{G}/\mathcal{H}$  by Lemma A.5. Therefore, there exists a  $\mathcal{H}$ -invariant vector  $\xi \in H$  of norm one such that  $\|\pi(g)\xi - \xi\| < \varepsilon$  for all  $[g] \in \psi(\mathcal{Q})$ . Since  $\xi$  is  $\mathcal{H}$ -invariant, it is also  $\mathcal{K}$ -invariant. Moreover, by definition of the map  $\psi$ , we have that  $\|\pi(g)\xi - \xi\| < \varepsilon$  for all  $[g] \in \mathcal{Q}$ . Thus,  $\pi$  has  $\mathcal{K}$ -invariant almost  $\mathcal{G}$ -invariant vectors.

The second claim of the proposition follows by taking  $\mathcal{K} := \{e\}$ .  $\square$

**Proposition A.6.** *Let  $\mathcal{G}$  be a topological group and  $\mathcal{H}$  a compact subgroup. Then  $\mathcal{G}$  has property (T) if and only if the pair  $(\mathcal{G}, \mathcal{H})$  has property (T).*

**Proof:** The left to right direction follows directly from Proposition A.4.

For the other implication, assume that the pair  $(\mathcal{G}, \mathcal{H})$  has property (T). Let  $\pi : \mathcal{G} \rightarrow \mathcal{U}(H)$  be a unitary representation that has almost  $\mathcal{G}$ -invariant vectors. We must prove that  $\pi$  has nontrivial  $\mathcal{G}$ -invariant vectors. This will follow immediately from property (T) of  $(\mathcal{G}, \mathcal{H})$  if we can prove that  $\pi$  has  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors.

Let  $\mathcal{Q} \subseteq \mathcal{G}/\mathcal{H}$  be a compact set and let  $\varepsilon > 0$ . We can assume without loss of generality that  $\varepsilon < 1$ . Let  $q$  denote the canonical map  $q : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ . Since  $\mathcal{H}$  is compact, we have that  $q^{-1}(\mathcal{Q})$  is a compact subset of  $\mathcal{G}$  and therefore so is  $q^{-1}(\mathcal{Q}) \cup \mathcal{H}$ . Hence, there exists a vector  $\xi \in H$  of norm 1 such that  $\|\pi(g)\xi - \xi\| < \varepsilon/4$  for all  $g \in q^{-1}(\mathcal{Q}) \cup \mathcal{H}$ .

Let  $\mu$  be the normalized Haar measure of  $\mathcal{H}$  such that  $\mu(\mathcal{H}) = 1$  and let  $\eta \in H$  be the vector  $\eta := \int_{\mathcal{H}} \pi(h)\xi \, d\mu(h)$ . We have that

$$\|\eta - \xi\| = \left\| \int_{\mathcal{H}} \pi(h)\xi - \xi \, d\mu(h) \right\| \leq \sup_{h \in \mathcal{H}} \|\pi(h)\xi - \xi\| < \varepsilon/4,$$

where we used the fact that  $\mathcal{H} \subseteq q^{-1}(\mathcal{Q}) \cup \mathcal{H}$ . Since  $\xi$  has norm one and  $\varepsilon$  is assumed to be less than 1, it follows that  $\eta$  is nonzero. Thus,  $\eta_0 := \frac{1}{\|\eta\|}\eta$  is an  $\mathcal{H}$ -invariant unit vector. Note that

$$\|\eta\| \geq \|\xi\| - \|\eta - \xi\| > 1 - \varepsilon/4 > 1/2,$$

again using that  $\varepsilon < 1$ . Hence, for every  $g \in q^{-1}(\mathcal{Q}) \cup \mathcal{H}$  we have

$$\begin{aligned} \|\pi(g)\eta_0 - \eta_0\| &= \frac{1}{\|\eta\|} \left\| \int_{\mathcal{H}} \pi(g)\pi(h)\xi - \pi(h)\xi \, d\mu(h) \right\| \\ &< 2 \left\| \int_{\mathcal{H}} \pi(g)\pi(h)\xi - \pi(h)\xi \, d\mu(h) \right\| \\ &\leq 2 \sup_{h \in \mathcal{H}} \|\pi(g)\pi(h)\xi - \pi(h)\xi\| \\ &\leq 2 \sup_{h \in \mathcal{H}} (\|\pi(g)\pi(h)\xi - \xi\| + \|\pi(h)\xi - \xi\|). \end{aligned}$$

Since  $g \in q^{-1}(\mathcal{Q}) \cup \mathcal{H}$  it follows that  $gh \in q^{-1}(\mathcal{Q}) \cup \mathcal{H}$  for any  $h \in \mathcal{H}$ . Thus, we have that

$$\|\pi(g)\eta_0 - \eta_0\| < 2(\varepsilon/4 + \varepsilon/4) = \varepsilon.$$

In particular, this means that  $\|\pi(g)\eta_0 - \eta_0\| < \varepsilon$  for all  $[g] \in \mathcal{Q}$ . We have thus proved that  $\pi$  has  $\mathcal{H}$ -invariant almost  $\mathcal{G}$ -invariant vectors.  $\square$

We are finally ready to verify Tzanev's claim that for a discrete Hecke pair  $(G, \Gamma)$ , property (T) from Definition A.1 is equivalent to property (T) for the topological group  $\overline{G}$  in the Schlichting completion  $(\overline{G}, \overline{\Gamma})$ . Recall that  $(G, \Gamma)$  is *reduced* if  $R^\Gamma = \cap_{g \in G} g\Gamma g^{-1}$  is the trivial subgroup  $\{e\}$ . For an arbitrary Hecke pair  $(G, \Gamma)$ , the reduction  $(G_r, \Gamma_r) := (G/R^\Gamma, \Gamma/R^\Gamma)$  is a reduced Hecke pair, and there are canonical isomorphisms between  $\mathcal{H}(G, \Gamma)$ ,  $\mathcal{H}(G_r, \Gamma_r)$  and  $\mathcal{H}(\overline{G}, \overline{\Gamma})$ , see [15] and [7].

**Theorem A.7.** *Let  $(G, \Gamma)$  be a Hecke pair. The following are equivalent:*

- i) *The pair  $(G, \Gamma)$  has property (T).*
- ii) *The pair  $(G_r, \Gamma_r)$  has property (T).*
- iii) *The pair  $(\overline{G}, \overline{\Gamma})$  has property (T).*
- iv) *The group  $\overline{G}$  has property (T).*

Here  $G$  and  $G_r$  are assumed to have the discrete topology and  $\overline{G}$  is assumed to have its locally compact totally disconnected topology.

**Proof:**  $i) \iff ii)$  Follows directly from Proposition A.3 since  $R^\Gamma$  is a normal subgroup of  $G$  contained in  $\Gamma$ .

$iii) \iff iv)$  Follows directly from Proposition A.6.

$ii) \implies iii)$  Suppose that  $(G_r, \Gamma_r)$  has property (T). Let  $\pi : \overline{G} \rightarrow \mathcal{U}(H)$  be a unitary representation that has  $\overline{\Gamma}$ -invariant almost  $\overline{G}$ -invariant vectors. We claim that  $\pi$  has a true nontrivial  $\overline{G}$ -invariant vector.

Let  $\epsilon > 0$  and  $Q \subseteq G_r/\Gamma_r$  a finite set, say  $Q = \{g_1\Gamma_r, \dots, g_n\Gamma_r\}$ . Since  $\pi$  has  $\overline{\Gamma}$ -invariant almost  $\overline{G}$ -invariant vectors, there is a  $\overline{\Gamma}$ -invariant vector  $\xi \in H$  of norm one such that

$$\|\pi(g)\xi - \xi\| < \epsilon,$$

for all  $[g] \in \{g_1\overline{\Gamma}, \dots, g_n\overline{\Gamma}\}$ . By restriction, the same holds for all  $[g] \in \{g_1\Gamma_r, \dots, g_n\Gamma_r\}$ . Moreover, being  $\overline{\Gamma}$ -invariant, the vector  $\xi$  is  $\Gamma_r$ -invariant. Hence, we showed that the restriction of  $\pi$  to  $G_r$  has  $\Gamma_r$ -invariant almost  $G_r$ -invariant vectors. By property (T) for  $(G_r, \Gamma_r)$  it follows that there exists a nontrivial  $G_r$ -invariant vector. By continuity of  $\pi$ , this vector must be  $\overline{G}$ -invariant.

$iii) \implies ii)$  Suppose  $(\overline{G}, \overline{\Gamma})$  has property (T). Let  $\pi : G_r \rightarrow \mathcal{U}(H)$  be a unitary representation that has  $\Gamma_r$ -invariant almost  $G_r$ -invariant vectors. We must show that it has a nontrivial  $G_r$ -invariant vector.

Let  $\mathcal{V} := \overline{\pi(G_r)H^{\Gamma_r}}$ , where  $H^{\Gamma_r}$  is the subspace of  $\Gamma_r$ -fixed vectors, which is nontrivial because  $\pi$  is assumed to have nontrivial  $\Gamma_r$ -invariant vectors. It is clear that  $\mathcal{V}$  is an invariant subspace for  $\pi$ , so the restriction of  $\pi$  to this subspace gives rise to a unitary representation  $\pi|_{\mathcal{V}}$  of  $G_r$  on  $\mathcal{V}$ . Moreover,  $\pi|_{\mathcal{V}}$  is clearly generated by the  $\Gamma_r$ -fixed vectors. By [7, Proposition 6.17], the representation  $\pi|_{\mathcal{V}}$  is continuous with respect to the Hecke topology, and therefore extends uniquely to a representation  $\tilde{\pi}|_{\mathcal{V}}$  of  $\overline{G}$  on  $\mathcal{V}$ , which is generated by its  $\overline{\Gamma}$ -fixed vectors.

Let  $\epsilon > 0$  and  $Q \subseteq \overline{G}/\overline{\Gamma}$  be a compact set. Since  $\overline{G}/\overline{\Gamma}$  is discrete,  $Q$  is in fact a finite set  $Q = \{g_1\overline{\Gamma}, \dots, g_n\overline{\Gamma}\}$ , where we can assume without loss of generality that the representatives  $g_1, \dots, g_n$  are elements of  $G_r$ , because of the canonical bijection between  $\overline{G}/\overline{\Gamma}$  and  $G_r/\Gamma_r$ . For



this chosen  $\varepsilon$  and the finite set  $\tilde{Q} := \{g_1\Gamma_r, \dots, g_n\Gamma_r\}$ , there exists a  $\Gamma_r$ -invariant unit vector  $\xi \in \mathcal{V}$  such that

$$\|\pi|(g)\xi - \xi\| < \varepsilon,$$

for every  $[g] \in \tilde{Q} \subseteq G_r/\Gamma_r$ . By continuity, the vector  $\xi$  is  $\bar{\Gamma}$ -invariant for  $\tilde{\pi}|$  and we have that  $\|\tilde{\pi}|(g)\xi - \xi\| < \varepsilon$  for every  $[g] \in Q$ . Hence,  $\tilde{\pi}|$  has  $\bar{\Gamma}$ -invariant almost  $\bar{G}$ -invariant vectors. By property (T) for  $(\bar{G}, \bar{\Gamma})$ , the representation  $\tilde{\pi}|$  has a nontrivial  $\bar{G}$ -invariant vector. Thus, by restricting  $\tilde{\pi}|$  to  $G_r$ , it follows that  $\pi$  has a nontrivial  $G_r$ -invariant vector.  $\square$

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